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COMMENT

The operator $p^2 - (ix)^\nu$ on $L^2(R)$ (reply to Comment by Bender and Wang)

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Abstract

The PT-invariant operator $H_\nu = p^2 - (ix)^\nu$ defined on $L^2(R)$ has a Hilbert–Schmidt resolvent for real $\nu > 1$, which is trace-class if $\nu > 2$ is not an integer divisible by 4. For $2 < \nu < 4$ the spectrum of H_ν coincides with the eigenvalues of the problem posed by Bender and Boettcher on a contour in the lower complex half-plane. We give the value of the spectral zeta function $Z(1; H_\nu)$ and investigate the limiting case $\nu \to +\infty$.

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Let $\nu > 0$ and define the function $U_\nu(x) = -(ix)^\nu$ (with $-3\pi/2 < \arg(x) < \pi/2$) in the complex plane cut along the positive imaginary half-axis for non-integer ν. Then, on the real axis the potential U_ν is PT-symmetric, $U_\nu(x) = \overline{U_\nu(-x)}$, but not separately invariant under reflection (P) and complex conjugation (T) (if $\nu \neq$ an even integer). Bender and Boettcher (1998) have introduced the operator

$$H_\nu = p^2 - (ix)^\nu \tag{1}$$

as a paradigmatic PT-invariant generalization of the cubic oscillator with imaginary coupling constant ($\nu = 3$). The latter had been conjectured by Bessis and Zinn-Justin (Bessis 1995) to have only positive eigenvalues. The connection between PT-invariance and the reality of the spectrum has been intensely investigated lately (Bender and Boettcher 1998, Bender *et al* 1999a,b, Bender *et al* 2000, Caliceti 2000, Delabaere and Pham 1999, Delabaere and Trinh 2000, Lévai and Znojil 2000, Mezincescu 2000, Shin 2001, Znojil 2000). Apparently, the reality of the spectrum is not quite robust under PT-invariant perturbations[2].

Bender and Boettcher (1998) have defined and studied a spectral problem for (1) by looking for solutions which decay on a contour connecting a pair of anti-Stokes lines in the

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[2] Examples of symmetry breaking for increasingly milder perturbations can be found the work of Bender and Boettcher (1998), Bender *et al* (1999b), Mezincescu (2000) and in the rather extensive references of Delabaere and Trinh (2000). The latter have shown that complex eigenvalues appear in the spectrum of $p^2 + ix^3 + i\alpha x$ for sufficiently large negative α.

lower complex half-plane [$\arg(z) = -\pi(\nu-2)/(2\nu+4)$ and $\arg(z) = -\pi + (\nu-2)/(2\nu+4)$] without crossing the cut. For $2 \leqslant \nu < 4$ the ends of this contour are in the same Stokes sectors as the two real half-axes, so that it can be deformed continuously into the real axis. Thus, for $2 \leqslant \nu < 4$ the Bender and Boettcher eigenvalues will coincide with those obtained by an orthodox approach on $L^2(R)$. For larger ν, the connection with that approach is not clear. While the contour procedure yields spectra, the development of an orthodox quantum mechanical apparatus requires the definition of a suitable space of states, on which the observables should act. Despite some progress in that direction (Bender and Boettcher 1998, Bender *et al* 1999b, Bender *et al* 2000), it is still unclear what that space of states will be.

In their Comment, Bender and Wang (2001) present a formula for the sum of the inverse eigenvalues (the spectral zeta function $Z(1; H_\nu)$) of the Bender–Boettcher spectral problem for $\nu > 2$:

$$Z(1; H_\nu) = A\frac{\Gamma\left(\frac{1}{\nu+2}\right)\Gamma\left(\frac{2}{\nu+2}\right)\Gamma\left(\frac{\nu-2}{\nu+2}\right)}{(\nu+2)^{\frac{2\nu}{\nu+2}}\Gamma\left(\frac{\nu-1}{\nu+2}\right)\Gamma\left(\frac{\nu}{\nu+2}\right)} \tag{2}$$

where

$$A = A_{\mathrm{BW}} = 4\sin^2\left(\frac{\pi}{\nu+2}\right) \tag{3}$$

and find excellent agreement with the results of several numerical and WKB calculations.

In this note, I give the corresponding results for the operator defined by (1) on the domain intersection $D(H_\nu) = D(p^2) \cap D(|x|^\nu) \subset L^2(R)$ for $\nu > 2$.

Repeating for non-integer ν the arguments in Caliceti *et al* (1980) shows that H_ν is closed and has a compact, Hilbert–Schmidt resolvent for $\nu \geqslant 1$, $\nu \neq 4M$. If ν is an integer divisible by 4, then H_{4M} is symmetric (Hermitian), but not self-adjoint[3], with deficiency indices (2, 2). In this case, there is an infinite (four-parameter) family of self-adjoint extensions. All have Hilbert–Schmidt resolvents (see, e.g., Caliceti *et al* 1980).

If ν is a natural number, the solutions of the differential equation

$$(z - H_\nu)f = f'' + [z + (\mathrm{i}x)^\nu]f = 0 \tag{4}$$

are entire functions. This has been exploited recently by Shin (2001) for $\nu = 3$ to obtain bounds on the location of the eigenvalues and the position of the zeros of the eigenfunctions. For general ν, by Cauchy's theorem on linear equations the solutions of (4) are holomorphic in the cut plane. Since the point $x = 0$ is a regular singularity of the equation, the solutions and their first derivatives will be continuous on the real axis. For $\nu \neq 4M$, $f_\pm(x, z)$—the 'Jost solutions' of (4) which are locally L^2, together with their first derivatives, near $\pm\infty$—are unique (up to arbitrary multiplicative factors). At zero energy ($z = 0$), the 'Jost solutions' can be given explicitly,

$$f_+(x; 0) = x^{1/2}K_\mu\left[2\mu \mathrm{e}^{\mathrm{i}\chi(\nu)}x^{1/2\mu}\right]. \tag{5}$$

Here, K_μ is Macdonald's function,

$$\mu = \frac{1}{\nu+2} \qquad \text{and} \qquad \chi(\nu) = \left[\mathrm{frac}\left(\frac{\nu}{4}\right) - \frac{1}{2}\right]\pi. \tag{6}$$

The integer and fractionary parts of the real number α are denoted by $\mathrm{int}(\alpha)$ and $\mathrm{frac}(\alpha)$. The angle $\chi(\nu)$, defined in (6) is continuous for $\nu \neq 4M$. The values of the phase factor $\mathrm{e}^{\mathrm{i}\chi(\nu)}$ lie in the right half-plane, tending to $\pm\mathrm{i}$ as $\nu \to 4M \mp 0$. When going from positive x to negative

[3] In plain language: if $\nu \neq 4M$, then $\mathrm{Im}(U_\nu) \neq 0$ and the closure of the domain $D(H_\nu)$ cannot contain elements with finite 'total energy', but infinite 'kinetic energy'. For $\nu = 4M$, the imaginary part of the potential is zero, while the real part is negative. The closure of $D(H_\nu)$ contains such elements and the enclosure $D(H_\nu^*) \subset D(H_\nu)$ is *strict*. For $\nu = 4M + 2$ the potential is again real, *but positive*, and H_{4M+2} is essentially self-adjoint.

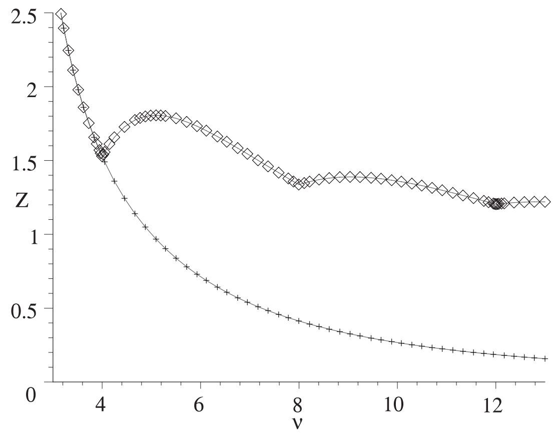

Figure 1. The ν dependence of $Z(1; H_\nu)$ from equation (2), with the factor A given by: crosses, equation (3) and diamonds, equation (7).

ones through the lower half-plane, the phase of the argument of K_μ in (5) goes from $\chi(\nu)$ to $-2\pi\,[1 + \mathrm{int}(\nu/4)] - \chi(\nu)$. One can show that $f_+(x; 0)$ *has no zeros on the real axis* for $\nu > 2$.[4] The function $f_-(x; 0)$ can be taken as the PT-conjugate of $f_+(x; 0)$.

Proceeding as in Mezincescu (2000), for $\nu \neq 4M$ the spectral zeta function is given again by (2), *but* with an additional term in the factor

$$A = A_{\mathrm{orthodox}} = A_{\mathrm{BW}} + 2\frac{\cos\left(\frac{4\chi(\nu)}{\nu+2}\right) - \cos\left(\frac{(\nu-2)\pi}{\nu+2}\right)}{1 + 2\cos\left(\frac{2\pi}{\nu+2}\right)}. \tag{7}$$

For $2 < \nu < 4$, the phase $\chi(\nu) = (\nu - 2)\pi/4$, and the additional term in (7) vanishes, as expected. Equation (7) is continuous for all $\nu > 2$, but has jumps in its derivative for $\nu = 4M$. In figure 1 we plot the ν dependence of (2) with A given by (3) and (7).

At the singular points, $\nu = 4M$, we can define two distinguished PT-invariant extensions H_{4M-0} and H_{4M+0} by the (norm) limits of the resolvent from the left and the right. Both satisfy (7).

For large $\nu \to +\infty$, the zeta function $Z_{\mathrm{orthodox}}(1; H_\nu)$ goes to the finite limit $\frac{2}{3}$ (equal to the value for a particle confined to $(-1, 1)$, while the Bender–Wang one vanishes as $2\pi^2/\nu^2$—the limiting value for their deformation (Bender *et al* 1999a). Actually, one can show that in the limit $\nu \to +\infty$, the resolvent kernel of the $L^2(R)$ operator H_ν tends *pointwise* to the resolvent kernel of the restriction of the second derivative to $L^2(-1, 1)$ with Dirichlet boundary conditions.

As a final remark, I am tempted to conjecture that the Bender–Boettcher deformation provides an analytic continuation of the eigenvalues from the interval $\nu \in (2, 4)$.

Note added in proof. I have just become aware of important work reported in two preprints by P Dorey, C Dunning and R Tateo (see Dorey *et al* (2000, 2001) and references therein). In particular, the 2001 preprint contains a proof of the reality of the spectrum of equation (1) and some generalizations.

[4] This allows us to factorize $H_\nu = -\left(\mathrm{d}/\mathrm{d}x + f'_+/f_+\right)\left(\mathrm{d}/\mathrm{d}x - f'_+/f_+\right)$. Then, the resolvent kernel is equal to the product of two upper/lower triangular operators. Both are Hilbert–Schmidt for $\nu > 2$. Thus, the resolvent kernel is *trace class*.

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